Stochastic Analysis of Interconnect Performance in the Presence of Process Variations

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Abstract

Deformations in interconnect due to process variations can lead to significant performance degradation in deep sub-micron circuits. Timing analyzers attempt to capture the effects of variation on delay with simplified models. The timing verification of RC or RLC networks requires the substitution of such simplified models with spatial stochastic processes that capture the random nature of process variations. The present work proposes a new and viable method to compute the stochastic response of interconnects. The technique models the stochastic response in an infinite dimensional Hilbert space in terms of orthogonal polynomial expansions. A finite representation is obtained by using the Galerkin approach of minimizing the Hilbert space norm of the residual error. The key advance of the proposed method is that it provides a functional representation of the response of the system in terms of the random variables that represent the process variations. The proposed algorithm has been implemented in a procedure called OPERA. Results from OPERA simulations on commercial design test cases match well with those from the classical Monte Carlo SPICE simulations and from perturbation methods. Additionally OPERA shows good computational efficiency: speedup factor of **60** *has been observed over Monte Carlo SPICE simulations.*

Introduction

The performance of integrated circuits (ICs) is increasingly less predictable as device dimensions shrink below the sub-100 nanometer scale. The modeling accuracy problem stems from poor control of the physical device and interconnect characteristics during the manufacturing process. Uncertainties due to variations in the manufacturing process are reflected in variations in the circuit parameters. Examples of manufacturing variations are the variations in materials, variations in geometry (t_{ox} L_{eff}, W) and doping profiles of MOSFETs, material and geometric variations of the interconnects etc. The many sources of variations in the IC fabrication process lead to a hierarchy of random and systematic effects on circuit performance [16].

A common way of accounting for process variations is to use a linear model to represent a circuit parameter. For instance, a parameter p would be expressed as $p = \mu_p + \epsilon_{1,p} + \epsilon_{2,p}$, where μ_p is a (nominal) mean value, $\epsilon_{1,p}$ and $\epsilon_{2,p}$ are random variables with mean zero and variances $\sigma_{1,p}$ and $\sigma_{2,p}$. These represent the inter-die and intra-die variations, respectively. Designers interested in performance analysis and optimization typically use only a single value of σ_p.

The performance of devices and interconnects depends on several parameters (L_{eff}, W, t_{ox}, V_t, etc.) that are *spatially* correlated across a chip and have systematic variations between otherwise identical dies [2]. Although the importance of accounting for correlations has been repeatedly emphasized, in practice the variational components of different parameters have been modeled mostly as independent random variables. Assuming independence of parameters leads to the analysis of extremely unlikely or even physically impossible circuits [17]. When correlation data or models are available, it is possible to generate values of multiple parameters by Monte Carlo sampling from their *joint* distribution functions [4, 3]. Joint distributions are usually not available and sample sizes need to be very large for modern VLSI designs. Perturbation techniques [8, 9, 15, 23] are an alternative but they are generally applicable to small variations (about the nominal values) and expansions beyond the first or second order may be computationally not feasible.

In the nanometer regime, there is a need for accurate models of interconnect that account for the *uncertainty* resulting from process variations [22]. For this reason, interconnect variational analysis based on model order reduction has been a very active topic of research over the past several years. Liu et. al. [15] studied the effect of interconnect parameter variations on three projection-based model order reduction techniques: Krylov subspace analysis methods [20, 18], PACT [12] and PRIMA [18]. The work combines matrix perturbation theory and the model order reduction methods. The authors of [9] proposed a balanced truncation (BTR) method for analysis of interconnects that accommodates variations. It offers a weighted error bound. The methods in [9] and [15] directly approximate the projection matrices as perturbed matrices from the nominal ones. The reduced system is unable to preserve stability. As a result, subsequent analysis with nonlinear devices can cause instability [14]. In [23], a BTR-like method using linear fractional transforms is described. It models variations but preserves stability and passivity. The new models have computable error bounds and unlike the existing variational analysis methods, impose no constraints on the internal structure of the state-space model.

Contributions of this work

This paper presents a new approach for the performance analysis of interconnect networks in the presence of process variations. The approach treats the electrical parameters in the system of differential equations for RC/RLC networks as continuous parameter (spatial) stochastic processes, where the source of randomness is due to variations in the parameters. As a result, the system response is a stochastic process which we show can be represented as a infinite series of orthogonal polynomials in a Hilbert space of random variables. The series is truncated by projecting it onto a finite dimensional space, while minimizing the error. This provides a functional or analytic representation of the stochastic process that includes the random parameters. With this, there is no need to repeatedly solve the system with values of the parameters as would be required in a Monte Carlo approach. Rather, the functional representation can be directly evaluated. Moments and probability density functions of quantities of interest may easily be computed from the resulting analytical from. Much higher order expansions are possible when compared to perturbation techniques. The procedure improves the computational efficiency of the Monte Carlo approach by an order of magnitude. The proposed technique has been implemented in a prototype software named OPERA (Orthogonal Polynomial Expansion for Response Analysis) which can also carry out a SPICE Monte Carlo analysis.

Problem Definition

Consider an interconnect that is represented as a RC network[1]. In the s-domain, it is described by the MNA equations as $(G + sC)x(s) = f(s)$, where $f(s)$ is the known input, $M(s) = (G + sC)$ is the coefficient matrix and $x(s)$ is the response to be determined. The circuit parameters G and C depend on the interconnect geometry, such as the metal height (H), metal width (W), ILD thickness (T), etc. In the presence of process variations, these geometric characteristics of the interconnect, and consequently, its electrical characteristics, must be modeled as random variables.

Suppose that there are r geometric characteristics of interest. Let Ω denote the sample space of experimental or *manufacturing outcomes*. For $\omega \in \Omega$, let $\vec{\xi}(\omega) = \{\xi_1(\omega), \ldots \xi_r(\omega)\}$ be a random variable that represents the value of the r geometric characteristics. The space of all such random variables is denoted by $\Theta : \Omega \to R^r$. Without loss of generality, we assume that random variables in $\vec{\xi}$ have zero mean. In the presence of process variations, the MNA equations take the form

$$M(s, \vec{\xi}(\omega))\ x(s, \vec{\xi}(\omega)) = f(s, \vec{\xi}(\omega)) \qquad (1)$$

where $M(s, \vec{\xi}(\omega)) = G(\vec{\xi}(\omega)) + sC(\vec{\xi}(\omega))$

In Equation (1), the coefficient matrix $M(s, \vec{\xi}(\omega))$ is a random process, representing the fact that the uncertainty or randomness in the system is in the system parameters. Because of this, the response $x(s, \vec{\xi}(\omega))$ is also a random process. $f(s, \vec{\xi}(\omega))$ includes the deterministic input as well as the random parameters. The domain of the index variable s is the complex domain. For fixed s, $x(s, \vec{\xi}(\omega))$ is a random variable. That is for each manufacturing outcome ω, and the corresponding value of the observed parameter $\xi(\omega)$, $x(s, \vec{\xi}(\omega))$ would be the response of the system for that specific manufacturing outcome. For a fixed $\vec{\xi}(\omega)$, $x(s, \vec{\xi}(\omega))$ is a deterministic function of s. Next, we present our method to compute the stochastic response $x(s, \vec{\xi}(\omega))$.

Approach

Overview

We assume that the stochastic response $x(s, \vec{\xi}(w))$ is a second order process. This means that all the random variables have finite variance. This is certainly valid for all stochastic models of real systems. The approach presented in this paper is based on expanding a stochastic process as an infinite series of orthogonal polynomials involving an arbitrary number of random variables. For reasons that will be made clear shortly, these polynomials are known as *polynomial chaos* [6]. Specifically, the stochastic response $x(s, \vec{\xi}(w))$ will be represented as

$$x(s, \vec{\xi}(\omega)) = \sum_{i=0}^{\infty} \alpha_i(s)\ \Psi_i(\vec{\xi}(\omega)) \qquad (2)$$

where $\{\alpha_i(s)\}$ are *deterministic* coefficient functions, $\vec{\xi}(\omega)$ are orthonormal random variables and $\Psi(\vec{\xi}(\omega))$ are a collection of multi-dimensional orthogonal polynomials in the random variables $\vec{\xi}(\omega)$. The equality in Equation (2) represents convergence in the norm. As will be explained shortly, the $\{\Psi_k\}$ constitute a orthonormal basis of a infinite dimensional Hilbert space, and there is a considerable choice in the selection of that basis. However, as long as the process is second order, convergence is guaranteed by any one of those orthonormal bases [5].

[1] The proposed method and results are for general RLC networks.

Orthogonal Expansions

In this section we explain how the representation of a random process given in Equation (2) is obtained. For simplicity, we will henceforth no longer explicitly show the dependence of ξ (and other random variables) on ω. For each s, $M(s, \vec{\xi})$ is a functional. That is, $M(s, \vec{\xi})$ maps each function $\vec{\xi} \in \Theta$ to a point in complex domain. Hence for each s, the response $x(s, \vec{\xi})$ is a point in the space Θ. The space Θ is an infinite dimensional function space. It is for this reason that the expansion has meaning in a Hilbert space. Once such a representation is obtained, we find the best finite approximation to the infinite series expansion of the process. The *best* is in terms of the underlying norm defined in the Hilbert space. In this section we provide only a few essential definitions and main results. Details of this theory are may be found in [6, 26].

Hilbert spaces provide a means for defining a inner product on a space of random variables (e.g. Θ). This leads to a norm and a metric, which in turn can be used to define convergence when representing a random variable as an infinite series. Convergence is in the mean square sense.

Definition 1 *Let $\mathcal{H}$ be a vector space over some field $\mathcal{F}$ with an inner product $< \cdot, \cdot >$ defined. The norm in $\mathcal{H}$ is $\|f\| = \sqrt{< f, f >}$, and the metric is $d(f, g) = \|f - g\|$. $\mathcal{H}$ is called a Hilbert space if it is* complete *as a metric space.*

Completeness means that if all terms of a sequence beyond a given point get arbitrarily close to each other (i.e. if it is a Cauchy sequence) then the sequence will converge. This allows determining convergence without knowing the limit.

Definition 2 (Orthogonal) *Two elements, x and y of an inner product space are said to be orthogonal if $\langle x, y\rangle = 0$. If in addition, $\|x\| = \|y\| = 1$, they are orthonormal.*

Definition 3 (Orthonormal Basis) *An* orthonormal sequence *$\{\phi_k\}_{k=1}^{\infty}$, in a Hilbert space is called an* orthonormal basis *if the only element outside the basis that is orthogonal to every element in the basis is the zero element. That is, an orthonormal basis is a* maximal *subset of elements that are mutually orthogonal.*

Lemma 1 *Let $\{\phi_k\}_{k=1}^{\infty}$ be an orthonormal basis of a Hilbert space. Then the infinite series $\sum_{k=1}^{\infty} \langle x, \phi_k\rangle\, \phi_k$ converges in norm to x [26].*

The space Θ is an infinite dimensional Hilbert space. The above lemma states that in order to obtain a convergent infinite series representation of a element in Θ, we need to find an orthonormal basis. One such basis is the set of Hermite polynomials. In Θ the inner product of any two elements is the expectation of their product, i.e., their correlation. Let P be a probability measure on Ω. Then the inner product on Θ is defined as

$$\left\langle \vec{\xi}_1, \vec{\xi}_2 \right\rangle = \mathbf{E}(\vec{\xi}_1, \vec{\xi}_2) = \int_{\Omega} \vec{\xi}_1 \vec{\xi}_2 dP \qquad (3)$$

Definition 4 (Hermite Polynomial) *Let $\xi_1, \xi_2, \ldots,$ be a infinite collection of variables. The Hermite polynomial of order p is defined as*

$$H_p(\{i_1, i_2, \cdots, i_p\}) = (-1)^p e^{\frac{1}{2}\vec{\xi}^t\vec{\xi}} \frac{\partial^p}{\partial\xi_{i_1}\partial\xi_{i_2}\cdots\partial\xi_{i_p}} e^{-\frac{1}{2}\vec{\xi}^t\vec{\xi}} \qquad (4)$$

where $\vec{\xi} = [\xi_{i_1}\ \xi_{i_2}\ \ldots\xi_{i_p}]^t$.

Note: In $H_p(\{i_1, i_2, \cdots, i_p\})$, any choice of p variables from $\{\xi_1, \xi_2, \ldots, \infty\}$ is allowed, including repetitions. If there are r variables (r dimensional), then there will be $(p + r - 1)!/p!(r - 1)!$ Hermite polynomials of degree p. As an example, consider two variables: $\vec{\xi} = \{\xi_1, \xi_2\}$. The

Hermite polynomials shown below are of order 0, 1, 2, and 3, and are easily derived using Equation (4).

$$\begin{aligned}
&\text{order 0:} \quad H_0(\{\}) = 1, \\
&\text{order 1:} \quad H_1(1) = \xi_1, H_1(2) = \xi_2 \\
&\text{order 2:} \quad H_2(1,1) = \xi_1^2 - 1, \ H_2(1,2) = \xi_1\xi_2, \\
&\qquad\qquad H_2(2,2) = \xi_2^2 - 1 \\
&\text{order 3:} \quad H_3(1,1,1) = \xi_1^3 - 3\xi_1, \\
&\qquad\qquad H_3(2,1,1) = \xi_1^2\xi_2 - \xi_2, \\
&\qquad\qquad H_3(2,2,1) = \xi_1\xi_2^2 - \xi_1, \\
&\qquad\qquad H_3(2,2,2) = \xi_2^3 - 3\xi_2
\end{aligned} \tag{5}$$

Let $\xi_1, \xi_2, \ldots$ denote a infinite set of zero mean, orthonormal **Gaussian** random variables. The (infinite) set of Hermite polynomials (of all orders) over the set of variables $\xi_1, \xi_2, \ldots$, constitute an orthonormal basis for Θ. This means that **any** second order stochastic process $x(s, \vec{\xi})$ can be expanded as an infinite series of Hermite polynomials over an infinite collection of zero mean, orthonormal Gaussian random variables. The expansion of stochastic response of the interconnect network can be expressed as [6]

$$\begin{aligned}
x(s,\vec{\xi}) &= c_0(s) H_0 + \sum_{i_1=1}^{\infty} c_{i_1}(s)\ H_1(\xi_{i_1}) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} c_{i_1 i_2}(s)\ H_2(\xi_{i_1},\xi_{i_2}) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3}(s)\ H_3(\xi_{i_1},\xi_{i_2},\xi_{i_3}) \\
&\vdots
\end{aligned} \tag{6}$$

Note that Equation (2) is merely a relabeling of the coefficients in Equation (6). To see this, we evaluate Equation (6) for $\vec{\xi} = \{\xi_1, \xi_2\}$ for up to order *three* using the polynomials already computed in Equation (5).

$$\begin{aligned}
x(s,\xi_1,\xi_2) &= c_0(s) + c_1(s)\xi_1 + c_2(s)\xi_2 \\
&+ c_{11}(s)(\xi_1^2 - 1) + c_{21}(s)(\xi_1\xi_2) + c_{22}(s)(\xi_2^2 - 1) \\
&+ c_{111}(s)(\xi_1^3 - 3\xi_1) + c_{211}(s)(\xi_1^2\xi_2 - \xi_2) \\
&+ c_{221}(s)(\xi_1\xi_2^2 - \xi_1) + c_{222}(s)(\xi_2^3 - 3\xi_2)
\end{aligned} \tag{7}$$

The one-to-one correspondence between the terms in Equation (2) and Equation (7) is clear.

The expansion in Equation (6) was first proved by Wiener [25] for a *Brownian Motion* process, and was known as *Homogeneous Chaos*. The result from [5] implies that the expansion is valid for any second order stochastic process, and it is now referred to as *polynomial chaos*.

The fact that $\{\xi_i\}_{i=1}^{\infty}$ are Gaussian is not a restriction. The original random variables $\vec{\xi}$ representing the interconnect variations can be Gaussian or non-Gaussian. The expansion using Hermite polynomials is still correct. If the parameter variations happen to be Gaussian, then the convergence will be exponentially fast [27].

The inner product on Θ given by Equation (3) is an expectation, and hence involves the probability measure P. Let $dP(\vec{\xi}) = w(\vec{\xi})d\vec{\xi}$. Then from Equation (3), we see that the inner product varies with the choice of the density or weight function $w(\vec{\xi})$. In the case that the interconnect variations represented by $\vec{\xi}$ are Gaussian, then the *optimal* choice (w.r.t to speed of convergence) for $w(\vec{\xi})$ is the Gaussian density function. This will result in the Hermite polynomials being the orthonormal basis for the space Θ. Other common distributions lead to different orthogonal polynomials. Table 1 from [27] shows the *best* choice for the orthonormal basis for several probability densities. Several of the continuous densities might be alternative choices for interconnect parameter variations.

	Variable Distribution	Polynomial Class
Continuous	Gaussian	Hermite
	Log-normal	Hermite
	Gamma	Laguerre
	Beta	Jacobi
	Uniform	Legendre
Discrete	Poisson	Charlier
	Binomial	Krawtchouk
	Negative Binomial	Meixner
	Hypergeometric	Hahn

Table 1. Relationship between the distribution of random variables and the choice of orthogonal polynomials

Finite Approximation

The unknowns in the expansion of the stochastic response shown in Equation (2) are the deterministic coefficient functions $\alpha_i(s)$. The number of random variables in the expansion are finite. However, the polynomials are of all orders. We need to truncate the expansion after including only a finite number of terms. The criterion will be to minimize the error, and in doing so, we find the coefficients $\alpha_i(s)$.

The method to determine the coefficients is based on the well known principle of orthogonality. Suppose that we are dealing with two finite dimensional inner product spaces V of dimension n, and a subspace W of dimension m, $m < n$, and we wish to find the best approximation of a vector $v \in V$ by a vector $w \in W$. *Best* is in the sense of minimizing the norm of the error, i.e. $\| v - w \|$. The principle of orthogonality states that the best choice for w is the one that is orthogonal to the error $v - w$, i.e. determine the w such that $\langle v - w, w \rangle = 0$.

The extension of the principle of orthogonality for mapping a vector in a infinite dimensional inner product space to a finite dimensional subspace is known as the Galerkin method [6]. Let r be the dimensionality of the random variable $\vec{\xi}$, i.e., the number of random parameters of the system. Let $\tilde{x}_p(s, \vec{\xi})$ denote the truncated version of the response x, using only the first p order polynomials. That is

$$\tilde{x}_p(s,\vec{\xi}) = \sum_{i=0}^{N} \alpha_i(s)\Psi_i(\vec{\xi}) \tag{8}$$

$$\text{where } N = \sum_{k=0}^{p} {}_{(r-1+k)}C_k \tag{9}$$

Let Σ_p denote the error due to the truncation. It is simply the difference between right hand side of Equation (1) and the left hand side with $\tilde{x}_p$ replacing x. That is,

$$\Sigma_p(s,\vec{\xi}) = M(s,\vec{\xi})\tilde{x}_p(s,\vec{\xi}) - f(s,\vec{\xi}) \tag{10}$$

The orthogonality condition (inner product of the error and the truncated series must be zero) results in p equations that have to be solved for the coefficients. These equations are

$$\left\langle \Sigma_p(s,\vec{\xi}), \Psi_j(\vec{\xi}) \right\rangle = 0, \ \ j = 0,1,\ldots,N \tag{11}$$

An Example

We illustrate our approach with the aid of an example calculation on the circuit shown in Figure 1. The metal interconnect can be modeled as a second order RC circuit consisting of two RC sections. The input excitation is a constant voltage source V_n. The metal interconnect is subject to geometric process variations. Without loss of generality, we assume that the only variations of significance are in the width W and in thickness T of each section and that they have a Gaussian distribution. These are modeled as zero-mean normalized Gaussian random variables (ξ_{w_1}, ξ_{t_1} for RC section 1 and ξ_{w_2}, ξ_{t_2} for RC section 2). Thus for the two RC sections, we have

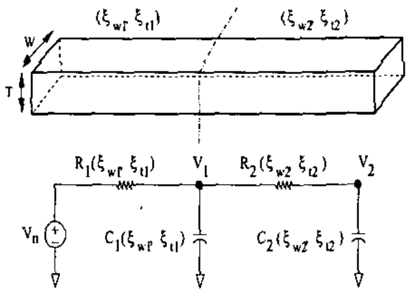

Figure 1. First Order RC Circuit

$$W_1 = W_{mean} + \sigma_{w_1}\xi_{w_1} \tag{12}$$
$$T_1 = T_{mean} + \sigma_{t_1}\xi_{t_1} \tag{13}$$
$$W_2 = W_{mean} + \sigma_{w_2}\xi_{w_2} \tag{14}$$
$$T_2 = T_{mean} + \sigma_{t_2}\xi_{t_2} \tag{15}$$

In general, the random variables ξ_{w_1}, ξ_{t_1}, ξ_{w_2}, ξ_{t_2} may be correlated. This implies that the resistance-conductance pairs of the RC sections are (implicit) functions of all the random variables. We thus have $\vec{\xi} = \{\xi_{w_1}, \xi_{t_1}, \xi_{w_2}, \xi_{t_2}\}$.

In this example, we attempt to capture the effects of the process variations on R_1 (or G_1), C_1 and R_2 (or G_2), C_2 by expressing them as a linear function of the geometric random variables. This is consistent with the models developed in much of the contemporary literature [15, 16]. However, we emphasize that there are no limitations in choosing any particular form of the expansion for G_1, G_2, C_1, C_2 in terms of $\vec{\xi}$. We thus have,

$$G_1(\vec{\xi}) = G_{M_1} + G_{W_1}\xi_{w_1} + G_{T_1}\xi_{t_1} \tag{16}$$
$$G_2(\vec{\xi}) = G_{M_2} + G_{W_2}\xi_{w_2} + G_{T_2}\xi_{t_2} \tag{17}$$
$$C_1(\vec{\xi}) = C_{M_1} + C_{W_1}\xi_{w_1} + C_{T_1}\xi_{t_1} \tag{18}$$
$$C_2(\vec{\xi}) = C_{M_2} + C_{W_2}\xi_{w_2} + C_{T_2}\xi_{t_2} \tag{19}$$

where G_{M_1} and G_{M_2} indicate the mean value of the conductances, and G_{W_1}, G_{W_2}, G_{T_1} and G_{T_2} signify the perturbations in G_1 and G_2 due to the variations in ξ_{w_1}, ξ_{w_2} and ξ_{t_1}, ξ_{t_2}. C_1 and C_2 are represented in the same way.

To make the illustration more tractable, and without loss of generality, we will assume that the variables ξ_{w_1} and ξ_{w_2} are same for the purposes of this analysis and so are ξ_{t_1} and ξ_{t_2}. Thus we have $\xi_{w_1} = \xi_{w_2} = \xi_1$ and $\xi_{t_1} = \xi_{t_2} = \xi_2$. We also assume that ξ_1 and ξ_2 are orthonormal. This is always possible to achieve by a linear transformation [19]. The MNA equation for our RC circuit is given by

$$(G(\vec{\xi}) + sC(\vec{\xi}))\,x(s,\vec{\xi}) = U(s)\,G_1(\vec{\xi}) \tag{20}$$

where

$$G(\vec{\xi}) = \begin{pmatrix} G_1(\vec{\xi}) + G_2(\vec{\xi}) & -G_2(\vec{\xi}) \\ -G_2(\vec{\xi}) & G_2(\vec{\xi}) \end{pmatrix} = G_a(s) + G_b(s)\,\xi_1 + G_c(s)\,\xi_2$$

$$C(\vec{\xi}) = \begin{pmatrix} C_1(\vec{\xi}) & 0 \\ 0 & C_2(\vec{\xi}) \end{pmatrix} = C_a(s) + C_b(s)\,\xi_1 + C_c(s)\,\xi_2$$

$$x(s,\vec{\xi}) = \left(V_1(s,\vec{\xi}), V_2(s,\vec{\xi})\right)^T,\ U(s) = (V_n, 0)^T$$

Matrices G_a, G_b, G_c and C_a, C_b, C_c are 2×2 symbolic matrices (they become numerical matrices for a specific set of real values of G_1, G_2, $C_1 C_2$).

We now expand the response $x(s,\vec{\xi})$ using second order ($p = 2$ in Equation (8)) Hermite polynomials.

$$x(s,\vec{\xi}) = \alpha_0(s) + \alpha_1(s)\,\xi_1 + \alpha_2(s)\,\xi_2 + \alpha_3(s)\,({\xi_1}^2 - 1) + \alpha_4(s)\,(\xi_1\xi_2) + \alpha_5(s)\,({\xi_2}^2 - 1) \tag{21}$$

Note that $\alpha_i(s)$ **is a two component vector** corresponding to each node in the network. That is, $\alpha_i(s) = (V_{1,i}(s), V_{2,i}(s))^T$

To obtain the response $x(s,\vec{\xi})$ we need to determine the coefficients $\alpha_i(s)$ using the Galerkin procedure described in Section . From (10) we have the definition of the error Σ_p as

$$\Sigma_p(s,\vec{\xi}) = (G(\vec{\xi}) + s\,C(\vec{\xi}))\,x(s,\vec{\xi}) - U(s)\,G_1(\vec{\xi}) \tag{22}$$

The coefficients $\alpha(s)$ are obtained by solving (see Equation (11))

$$\left\langle \Sigma_p(s,\vec{\xi}), \Psi_j(\vec{\xi}) \right\rangle = 0 \quad \text{for } j = 0, 1, \ldots, N \tag{23}$$

The inner product $\left\langle \Sigma_p(s,\vec{\xi}), \Psi_j \right\rangle$ is defined as

$$\left\langle \Sigma_p(s,\vec{\xi}), \Psi_j(\vec{\xi}) \right\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Sigma_p(s,\vec{\xi})\Psi_j(\vec{\xi})W(\vec{\xi})d\xi_1 d\xi_2 = 0 \tag{24}$$

where $w(\vec{\xi})$ is the standardized bivariate Gaussian probability density function.

Thus, for each $j = 0, 1, \cdots N$, Equation (23) gives us 2 equations in terms of the unknown deterministic coefficients represented by the vector $\alpha(s)$. This results in 12 linear equations with 12 unknowns in $\alpha(s)$. Expressing the linear system of equations obtained in a matrix form we have

$$(\tilde{G} + s\tilde{C})\,\alpha(s) = \tilde{b} \tag{25}$$

where

$$\tilde{G} = \begin{bmatrix} G_a & G_b & G_c & 0 & 0 & 0 \\ G_b & G_a & 0 & 2G_b & G_c & 0 \\ G_c & 0 & G_a & 0 & G_b & 2G_c \\ 0 & 2G_b & 0 & 2G_a & 0 & 0 \\ 0 & G_c & G_b & 0 & G_a & 0 \\ 0 & 0 & 2G_c & 0 & 0 & 2G_a \end{bmatrix} \tag{26}$$

The matrix $\tilde{C}$ has the same form as $\tilde{G}$. Finally $\tilde{b}$ is given by

$$\tilde{b} = (G_a(1,1)V_n(s), G_b(1,1)V_n(s), G_c(1,1)V_n(s), 0, 0, 0)^T \tag{27}$$

Now we can solve Equation (25) numerically to obtain the coefficient vector $\alpha(s)$. Once the vector $\alpha(s)$ is obtained, we have an explicit expression for the circuit response $x(s,\vec{\xi})$ in terms of $\vec{\xi}$ given by Equation (21). With this explicit expression the probability distribution of the delay from the source node to any node w.r.t to the geometric random variables $\vec{\xi}$ is readily determined. And some of the unique properties of the Hermite polynomials help us calculate easily the mean, variance, etc., of the delay distribution.

For a general RC circuit, if the dimensions of G, C and $U(s)$ are $k \times k$, $k \times k$ and $k \times 1$ respectively, then $\tilde{G}$, $\tilde{C}$ and $\tilde{b}$ are of the order $6k \times 6k$, $6k \times 6k$ and $6k \times 1$ respectively for a order 2 (p = 2) expansion of the stochastic circuit response.

General Method

Our approach for obtaining the stochastic response of an interconnect can be broadly classified into four steps:

Modeling the Interconnect system using stochastic MNA equations: Any interconnect can be modeled as an RC (or RLC) circuit with multiple π sections. An MNA equation similar to (1) can be obtained for every RC (or RLC) interconnect.

Expressing the Interconnect response as an infinite series of orthonormal basis set of polynomials: The stochastic interconnect response $(x(s, \vec{\xi}))$ can be expressed as an infinite series of an orthonormal basis set of polynomials using an expansion similar to that of (6). The choice of polynomial set depends on the probability distributions of the geometric random variables (Table)

Minimizing the error due to projection on to a finite subspace: The infinite series interconnect response expansion is truncated for an order p as shown in (8). To optimally minimize the error due to the finite truncation of the series, we minimize a norm of the error and each polynomial of the orthonormal basis set defined as in (11), (3).

Solving for the unknown coefficients of the finite series expansion: The coefficients of the finite series interconnect response expansion ($\alpha_j(s)$ in (8)) are deterministic and unknown. And the error norm minimization from step 3 gives us a linear system of equations in terms of these unknown coefficients ($\alpha_j(s)$). We can solve these equations numerically to obtain the stochastic interconnect response.

The computational steps of our approach described above have been implemented in a prototype software called OPERA. OPERA also has the additional ability to perform Monte Carlo SPICE simulations.

Computational Cost

The key computational steps of OPERA are evaluating the inner product in Equation (24) and solving Equation (25) for $\alpha(s)$. Since the integrands in the inner product are polynomials in $\vec{\xi}$ and include an exponential function $e^{\frac{1}{2}\xi^T\xi}$, integration by parts ensures easy numerical or even *symbolic* integration. In fact, for any given order p of the stochastic response expansion, the integration *need only be performed once* for symbolic values of matrices G and C followed by a substitution of the actual numerical matrices. The resulting symbolic block matrices $\tilde{G}$, $\tilde{C}$ and $\tilde{b}$ consist of some constant multiples of the sparse matrices G_a, G_b, G_c, C_a, C_b, C_c and V_n. In addition the matrices $\tilde{G}$ and $\tilde{C}$ have been observed to become increasingly sparse with an increase in the order p of the stochastic response expansion or an increase in the number of the uncertain parameters (random variables) of the interconnect circuit.

The computational cost of (25) increases linearly with the number of coefficients $\alpha(s)$. This depends on the order p of the expansion and the number of random variables r. If the order of the polynomial chaos is p, then the number of coefficients $\alpha(s)$ will be $O(r^p)$, where r is the number of random variables. The computational cost increases as a polynomial w.r.t to the number of uncertain parameters. To further reduce the computational complexity of OPERA model order reduction techniques are used.

Model Order Reduction

The model order reduction (MOR) techniques can be applied in two domains in our approach. Application of MOR in one domain targets the order p of the stochastic response expansion and in the other domain targets the number of terms approximating the coefficient $\alpha_j(s)$ in (8), (21). Both these techniques are integrated in to OPERA .

Stochastic Hilbert space domain: The order p of the stochastic response expansion in Equation (8) provides the first opportunity to limit the order of the system. However, the order of the system is primarily determined from the accuracy requirements and an order 2 or order 3 expansion seems to be adequate for practical purposes.

Deterministic Hilbert space domain: Equation (25) from section represents a deterministic system of linear equations of the unknown coefficients $\alpha(s)$. This is similar to the MNA equation of a generic RC (or RLC) interconnect in the absence of process variations. And hence all the existing MOR techniques can be applied to this system of equations.

Model order related stability issues have been discussed in a number of literatures and any existing stability technique can be applied in our method. The resulting system due to MOR will have almost a linear increase in complexity with regard to the number of uncertain parameters (random variables). Figure 2 shows the increase in the computational cost (for a sample interconnect) with increase in the number of uncertain parameters for the Monte Carlo SPICE simulations (SPMC) technique, OPERA without MOR, and OPERA with MOR. Cost for OPERA with MOR increases linearly with the number of the uncertain parameters.

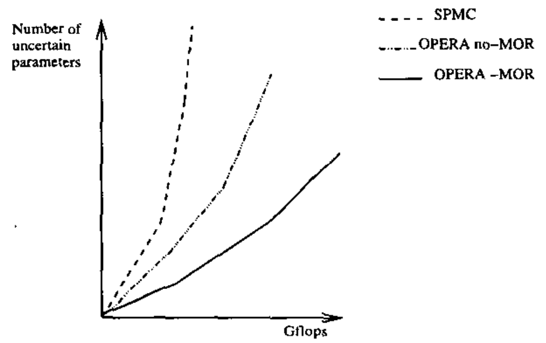

Figure 2. Computational cost increase trend

Experimental Results

OPERA has been verified for several test cases and the results for some representative test cases of RC and RLC interconnects are given below. The (normalized) random variables considered in all our test cases are width and thickness variations. Our algorithm has been verified for two cases of probability distributions of the random variables: 1) Gaussian, 2) Lognormal.

Gaussian Distribution

Case A. RC Tree: The first test case considered is an RC tree shown in Fig.3 ([11]). It is assumed that this RC tree

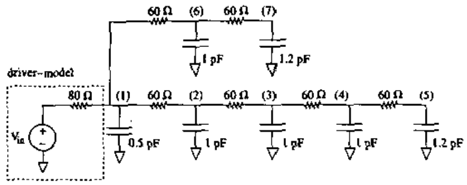

Figure 3. RC Tree

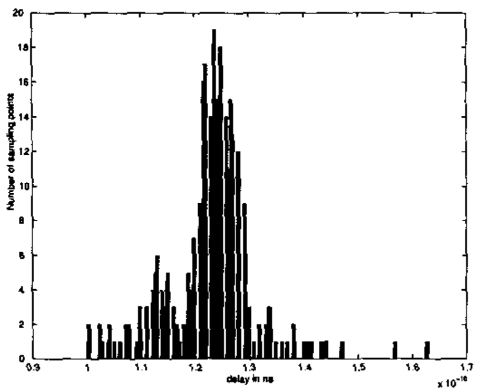

Figure 4. Delay distribution at a node with fixed wire thickness & changing wire width

is on Metal Layer 4 and is subject to metal wire width and thickness variations. We compare the delays obtained from SPICE based Monte Carlo (SPMC) simulations with 1000 sampling points with those from OPERA with an order 3 expansion. The results for the mean 50% delay and 90 % delay with a 3σ maximum width variation of 20% and thickness variation of 30% are summarized in Table 2. The differences between the delays obtained from OPERA and SPMC at each of the leaf nodes is about 0.1% or less.

Table 2. Comparison of SPMC and OPERA (time in ps)

Node	SPMC 50%	OPERA 50%	SPMC 90%	OPERA 90%
2	477.2	477.2	2021.5	2021.5
3	700.0	700.0	2272.5	2272.5
4	845.0	845.7	2423.4	2423.3
5	923.3	923.4	2511.2	2511.3
6	381.9	382.0	1792.8	1793.1
7	452.3	452.2	1821.0	1821.0

Figure 4 gives the distribution of delays at node 7 with regard to fixed wire thickness and changing wire width.

Case B. RLC Tree: Our approach is also applicable to RLC circuits. As an example, we consider a 2000 micron distributed RLC line in Fig. 5 ([11]), which is modeled with 20 lumped RLC sections. The results from OPERA with order 2 and order 3 expansions are compared with the results from SPMC in the frequency domain (Fig.6). The results from OPERA for an order 3 expansion match very well with those from SPMC. Thus for both RC and RLC circuits, OPERA with order 3 expansion offers good accuracy.

Case C. H-shaped Clock Tree: As a final example for the case of Gaussian distribution, we consider several large H-shaped clock trees (Fig.7) taken from a 0.13 micron commercial design. Table 3 shows the mean 50 % delay comparisons at a terminal sink node from OPERA and a 1000 sampling point SPMC for clock trees with a varying number of fanouts with a 3σ maximum width variation of 30% and thickness variation of 20 %. The time consumed by both the algorithms in flops (floating point operations) is listed in the table. It can be observed that an average speed up of about

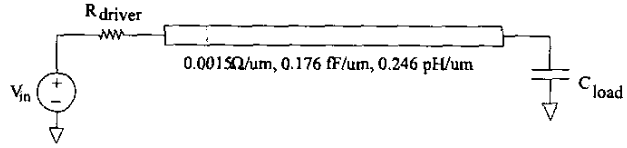

Figure 5. A distributed RLC line with a linear driver and capacitance load

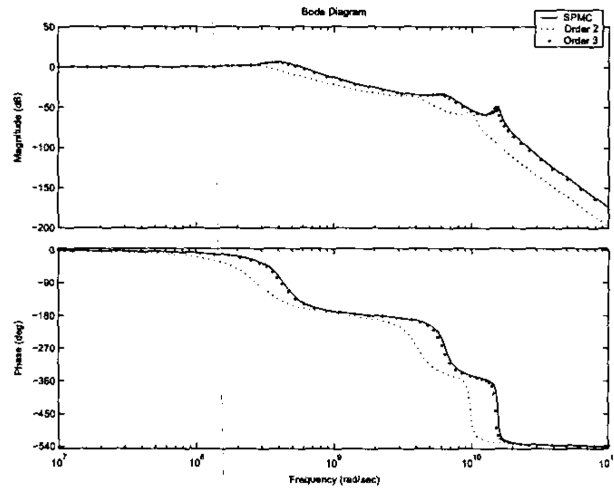

Figure 6. Frequency response with SPMC, order 2 and order 3 OPERA

60x is obtained by OPERA over SPMC.

And for a H-shaped clock tree, we obtained 50 $\%V_{dd}$ delay response at a sink node from OPERA and perturbation methods from [23, 15, 9] for different variations in the metal width and thickness. The comparison of the mean delay responses is shown in Table 4. For want of uptodate software versions for the perturbation methods from [23, 15, 9], we are unable to compare our timing complexity with those approaches.

Lognormal Distribution

Case D. RC Tree: The second probability distribution of the random variables we consider is a Lognormal distribution. For this distribution, we consider the case of an RC interconnect with 7 nodes. Modeling the conductance and the capacitance matrices in the presence of width and thickness variations that are Lognormal is the primary difficulty involved in this case. A Lognormal variable is defined as the exponential of a normal variable. Assuming that the variations in the normalized random variables width and thickness are small ($< e$), we recover the Gaussian case by performing a taylor series expansion of the exponential function of the Gaussian random variable. We truncate the exponential series to a required degree of accuracy, order two in this example. The rest of the procedure in obtaining the delay response is similar to the Gaussian distribution case. Table 5

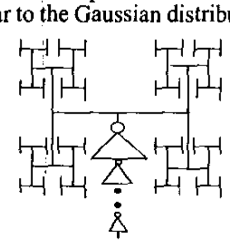

Figure 7. H-tree clock tree driven by a tapered buffer

Table 3. Comparison of SPMC and OPERA (time in ps)

# fan outs	SPMC 50% (ps)	OPERA 50% (ps)	error %	SPMC (Gflops)	OPERA (Gflops)	speed up
4	137.2	137.2	0.0	2.4	0.06	40
16	345.1	345.4	0.06	12.3	0.21	60
64	1284.9	1285.4	0.02	35.5	0.48	71
256	2263.3	2265.4	0.1	73.9	1.3	57
1024	4352.7	4353.4	0.01	148.4	2.3	60

Table 4. 50% V_{dd} delay of H shape clock tree at one sink

M4-w (%)	M4-h (%)	OPERA (ns)	SPMC (ns)	error (%)	[23] (ns)	[15] (ns)	[9] (ns)
-10.0	30.0	4.009	4.100	0.025	4.113	4.102	4.110
20.0	20.0	2.945	2.943	0.06	3.018	2.970	2.963
-5.0	20.0	3.502	3.505	0.01	3.489	3.505	3.521
-30.0	-30.0	4.990	4.990	0.00	4.983	4.980	4.990
30.0	-10.0	4.233	4.234	0.025	4.203	4.257	4.233
20.0	-5.0	3.420	3.425	0.01	3.442	3.454	3.440
4.0	4.0	3.447	3.448	0.08	3.447	3.448	3.448
10.0	-6.0	3.211	3.209	0.03	3.170	3.169	3.212
-20	10	2.833	2.830	0.02	2.890	2.890	2.830
25	-5	3.611	3.606	0.16	3.585	3.598	3.610
10	5	3.691	3.694	0.01	3.682	3.690	3.682

shows the comparison between the mean and standard deviation (σ) of 90% step delays obtained from OPERA (for an order 3 expansion) and from SPMC simulations (1000 sampling points) for each node of the RC Tree. A 3σ maximum width variation of 25 % and a thickness variation of 20 % were considered.

Table 5. Comparison of SPMC and OPERA (time in ns)

Node	SPMC Mean 90%	OPERA Mean 90%	SPMC σ 90%	OPERA σ 90%
2	17.97	17.95	1.160	1.180
3	22.43	22.41	1.419	1.451
4	25.42	25.40	1.591	1.620
5	27.47	27.45	1.705	1.730
6	28.82	28.80	1.770	1.791
7	29.59	29.57	1.805	1.835

Conclusions

We proposed a novel scheme for analyzing the performance of interconnects in the presence of process variations. The variations are modeled as random variables. We showed how the stochastic response of the interconnects can be efficiently computed by an infinite series orthonormal polynomial expansion of the response. This provides a novel framework for the development of sophisticated algorithms for accurate and precise stochastic model computations. We carried out simulations on sample test cases and test cases from commercial designs (0.13 micron technology). Comparison of our results using OPERA against the classical Monte Carlo based SPICE simulations demonstrates an excellent match. In addition, our algorithm demonstrates a significant speedup of the order of 60X over Monte Carlo SPICE simulations.

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