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DEPTH AND DELAY IN A NETWORK

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It is a widespread point of view that the delay of a combinational logical network is equal to its depth, i.e. to the maximum of the delays of its chains. It is clear that for any combinational network the delay is no larger than the depth. But it would be a mistake to assume that it cannot be smaller. The point is that in a network there can be chains (even with maximum delays) through which the signal never passes. This is trivial for redundant networks. In this paper we will show that minimal networks can also have this property. Therefore, the delay in a combinational logical network can be less than the depth, even if the network is minimal.

An example of such a network is given in Figure 1 ($x_1, \dots, x_7$ are input variables, $y_1, \dots, y_8$ variables at the outputs of the cells $E_1, \dots, E_8$). Let us denote the values of the variables at time t by $x_1(t), \dots, x_7(t), y_1(t), \dots, y_8(t)$ respectively. We will assume that while the network is active the values of the input variables remain unchanged. This means that at each time which interests us the following equality holds:

$$x_i(t)=x_i, \qquad i=1,\dots,7. \tag{1}$$

Further, we will assume that each cell has a delay equal to 1, i.e.

$$\begin{aligned} y_1(t) &= x_1(t-1) \vee x_2(t-1), \\ y_2(t) &= y_1(t-1)\, x_3(t-1), \\ &\cdots\cdots\cdots\cdots \\ y_8(t) &= y_5(t-1) \vee y_7(t-1). \end{aligned} \tag{2}$$

Under these conditions it is the longest chain $E_1, E_2, E_3, E_4, E_6, E_7, E_8$ that has the maximum delay. It is not difficult to see that this chain does not influence the output of the network. In fact, if $x_3 = 0$, then the cell E_2 is open, while if $x_3 = 1$, then the action of the chain is duplicated by the faster chain E_6, E_7, E_8. This intuitive argument can be formalized if, using (2) and (1), we make the following simple transformations:

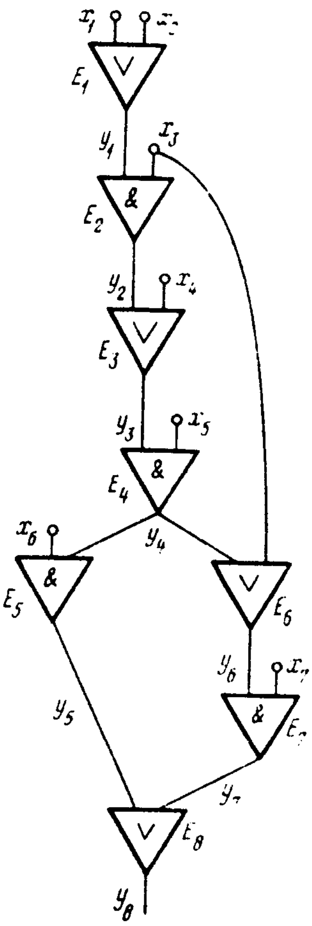

FIGURE 1

AMS (MOS) subject classifications (1970). Primary 94A20.

$$y_8(t)=y_5(t-1)\vee y_7(t-1)=y_4(t-2)x_6\vee y_6(t-2)x_7$$
$$=y_3(t-3)x_5x_6\vee(y_4(t-3)\vee x_3)x_7=(y_2(t-4)\vee x_4)x_5x_6\vee(y_3(t-4)x_5\vee x_3)x_7$$
$$=(y_1(t-5)x_3\vee x_4)x_5x_6\vee((y_2(t-5)\vee x_4)x_5\vee x_3)x_7$$
$$=((x_1\vee x_2)x_3\vee x_4)x_5x_6\vee((y_1(t-6)x_3\vee x_4)x_5\vee x_3)x_7.$$

Although $y_1(t-6)$ appears in the last line, in reality $y_8(t)$ does not depend on $y_1(t-6)$ (one can see this, for example, by expanding and simplifying the expression). It follows from this that the network given in Figure 1 has delay at most 6. At the same time, the delay of the longest chain $E_1, E_2, E_3, E_4, E_6, E_7, E_8$, and therefore the depth of the network, is 7.

It may seem strange that such a "nonworking" chain cannot be eliminated. The reason is that different parts of the chain belong to two different "working" chains: $E_1, E_2, E_3, E_4, E_5, E_8$ and E_6, E_7, E_8, and thus not one cell can be deleted from the "nonworking" chain without affecting the "working" chains.

We will now show that the network shown in Figure 1 is minimal (in the number of cells) in the class of networks constructed from &, $\vee$, $\neg$-cells. This network, as is easy to see, evaluates the function

$$\varphi_0(x_1,\ldots,x_7)=x_1x_3x_5x_6\vee x_2x_3x_5x_6\vee x_4x_5x_6\vee x_4x_5x_7\vee x_3x_7.$$

Let us denote by $L(f)$ the number of cells in a minimal network for an arbitrary function f. It is sufficient to show that

$$L(\varphi_0)\geqslant 8. \tag{3}$$

Consider any minimal network for the function φ_0. Since the function φ_0 depends essentially on the variable x_1, there is a cell in the network with input x_1. Putting $x_1=0$ and deleting from the network at least one cell (the cell with input x_1 and possibly the cells following it), we obtain a network for the function

$$\varphi_1(x_2,\ldots,x_7)=x_2x_3x_5x_6\vee x_4x_5x_6\vee x_4x_5x_7\vee x_3x_7.$$

From what has been said it is clear that

$$L(\varphi_0)\geqslant L(\varphi_1)+1. \tag{4}$$

Now consider any minimal network for the function φ_1. Obviously there is a cell in it with input x_2. If it is an &-cell or a $\neg$-cell, put $x_2=0$ and, deleting at least two cells (the cell with input x_2 and at least one cell following it), obtain a network for the function

$$\varphi_2(x_3,\ldots,x_7)=x_4x_5x_6\vee x_4x_5x_7\vee x_3x_7.$$

If it is an $\vee$-cell, put $x_2=1$ and, deleting at least two cells, obtain a network for the function

$$\varphi_3(x_3,\ldots,x_7)=x_3x_5x_6\vee x_4x_5x_6\vee x_4x_5x_7\vee x_3x_7.$$

In this way at least one of the following inequalities holds:

$$L(\varphi_1)\geqslant L(\varphi_2)+2,\quad L(\varphi_1)\geqslant L(\varphi_3)+2. \tag{5}$$

Next, put $x_4=1$ in a minimal network for the function φ_2 and in a network for the function φ_3. In either case, deleting at least one cell we obtain a network for the function

$$\varphi_4(x_3,x_5,x_6,x_7)=x_5x_6\vee x_5x_7\vee x_3x_7.$$

From this it follows that

$$L(\varphi_2)\geqslant L(\varphi_4)+1,\quad L(\varphi_3)\geqslant L(\varphi_4)+1.$$

Hence, regardless of which of the inequalities in (5) holds, the following is satisfied:

$$L(\varphi_1)\geqslant L(\varphi_4)+3. \tag{6}$$

Let us now look at any minimal network for the function φ_4 and a cell in it with input x_3. If it is an &-cell or a ¬-cell, put $x_3 = 0$ and, deleting at least two cells, obtain a network for the function

$$\varphi_5(x_5, x_6, x_7)=x_5x_6\vee x_5x_7.$$

If it is an ∨-cell, put $x_3 = 1$ and, deleting at least two cells, obtain a network for the function

$$\varphi_6(x_5, x_6, x_7)=x_5x_6\vee x_7.$$

Then one of the following inequalities holds:

$$L(\varphi_4)\geqslant L(\varphi_5)+2,\quad L(\varphi_4)\geqslant L(\varphi_6)+2. \tag{7}$$

The functions φ_5 and φ_6 depend essentially on all their variables. Hence,

$$L(\varphi_5)\geqslant 2,\quad L(\varphi_6)\geqslant 2.$$

Therefore, no matter which of the inequalities in (7) holds, the following bound is valid:

$$L(\varphi_4)\geqslant 4. \tag{8}$$

From (4), (6) and (8) we get (3). Thus, we have also proved that the network of Figure 1 is minimal and its delay is smaller than its depth.

Other minimal networks for the function φ_0 do not, however, possess this property, for example the network constructed in accordance with the formula

$$\varphi_0(x_1,\ldots,x_7)=((x_1\vee x_2\vee x_4)x_5x_6\vee x_7)(x_3\vee x_4x_5).$$

In connection with this the author has constructed a sequence of networks having stronger properties, which allows us to formulate the following claim: *for every natural l there exists a Boolean function f_l such that an arbitrary minimal network for f_l has delay $l + 8$ and depth $2l + 8$.*

Let us remark that for a smaller class of networks – those without branching cell outputs – the situation is different: the delay of any minimal (in this class) network is equal to the depth. This stems from the fact that in networks without branching outputs one can get rid of "nonworking" chains by deleting the cells whose signals take too long to reach the output of the network.

For this reason, even though delay does not coincide with depth in the general case, for specific Boolean functions the problem of minimizing delay coincides with the problem of minimizing depth. It is another matter if a network for a given Boolean function must satisfy some combination of conditions imposed on the number of its cells and its delay. In this case the distinction between delay and depths can play an important role. It is possible that at the expense of an increase in depth of a network (keeping its delay unchanged) one could decrease the number of its cells. The following example illustrates the likelihood of such a situation.

Consider a network of a parallel adder with acceleration carry circuits. Let $x_1, \ldots, x_n$ be the digits of the first summand (listed from the most significant), $y_1, \ldots, y_n$ the digits of the second summand, and let the digits in the construction of the carry circuits be partitioned into groups of k digits each.

We will describe the construction of a section of the network involving the most significant group (for other groups the corresponding sections of the network are constructed similarly). In the beginning of the calculation it computes secondary digits:

$$u_i = x_i y_i, \quad v_i = x_i \vee y_i, \quad 1 \leqslant i \leqslant k.$$

and at the end of the calculation, depending on the carry w_k going into a less significant digit of the group in the acceleration carry circuit, it computes carries for all the remaining digits of the group:

$$\begin{gathered} w_{k-1} = u_k \vee v_k w_k, \\ \cdots\cdots \\ w_1 = u_2 \vee v_2 w_2. \end{gathered} \tag{9}$$

Also, this section of the network computes two functions necessary for the construction of acceleration carry circuits:

$$v_1 v_2 \ldots v_k,$$

$$f(u_1, v_1, \ldots, u_k) = u_1 \vee v_1 (u_2 \vee v_2 (u_3 \vee \ldots (u_{k-1} \vee v_{k-1} u_k) \ldots)). \tag{10}$$

Let us see that happens if, for the computation of the function $f(u_1, v_1, \ldots, u_k)$, instead of (10) two such equalities are used:

$$w_0 = u_1 \vee v_1 w_1; \tag{11}$$

$$f(u_1, v_1, \ldots, u_k) = w_0 (u_1 \vee u_2 \vee \ldots \vee u_k) \tag{12}$$

((12) is easy to verify by consecutively getting rid of $w_0, \ldots, w_{k-1}$ using (11) and (9) and opening parentheses both in the resulting expression and in (1)). Let us turn our attention to the following peculiarity of (12): in spite of the fact that w_0 depends on w_k (see (11) and (9)), the function $f(u_1, v_1, \ldots, u_k)$ does not depend on w_k. Thanks to this, during the transformation of the network under consideration its delay hardly changes, while its depth grows sharply and becomes larger than $2n$. At the same time, the number of cells in the network decreases to $(k-4)n/k$, provided, of course, $k > 4$.

In this paper we have only considered combinational logical networks. The author, however, assumes that the distinction between delay and depth can also be observed in many other cybernetical networks. In other words, if a system of objects (events, etc.) has a sufficiently complex structure, then it is quite possible to trace a multi-level connection in it, leading from one object (event, etc.) to another, which, however, cannot for some reason or another exhibit itself. Some of these reasons have been indicated in considering the network of Figure 1.

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