Efficient model reduction of myelinated axons as port-Hamiltonian systems

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Summary. Modeling the myelinated axons in a realistic way, by maintaining the physical meaning of the components may lead to complex systems, described by high-dimensional systems of PDEs. The inclusion of myelinated axons into larger neuronal circuits requires the generation of equivalent low-order models that preserve the passivity and stability of the original models. The axons port-based network structure makes them suitable to be modeled as port-Hamiltonian systems. This paper uses a structure-preserving reduction method for port-Hamiltonian systems to reduce the global model of a myelinated axon.

1 Models of myelinated axons

A myelinated axon (Fig. 1) consists of myelinated sections through which the signal is transmitted, which alternate with Ranvier nodes where the signal is regenerated (saltatory conduction).

In order to model the transmission of signals through this chain, the phenomena occurring in the myelinated sections have to be coupled with the phenomena occurring in the Ranvier nodes. This coupling can be carried out by means of electric terminals defined both for the Ranvier nodes and for the myelinated sections [3], resulting in nonlinear PDEs. Consequently, in order to obtain a reduced model for the axon, both components (called nodes and internodes) can be modeled separately.

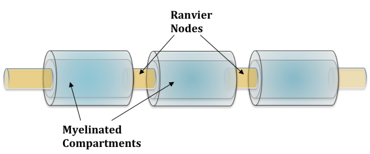

Fig. 1. The neuronal signal is transmitted along myelinated axons and regenerated in the Ranvier nodes.

The most popular approach to model the internodes is represented by the "cable model", described by 1D PDEs of parabolic type [5], namely the RC transmission line equation. A common reduction method for these models consists of discretizing the line into several segments, each being minimally modeled with lumped parameters (Fig. 2). The resulting numerical model is a network of RC sections having resistive parameters describing longitudinal electrical conduction phenomena through axoplasm, and capacitive and transverse conductive effects through the cell membrane.

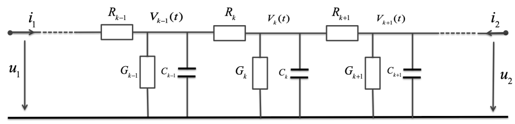

Fig. 2. The segmented model of an internode, a network of RC cells. The companion circuit is generated by the spatial discretization with centered differences of the transmission line equation.

2 Reduction of port-Hamiltonian systems

2.1 Port-Hamiltonian systems

This particular model of a myelinated compartment, described as an interconnection of RC cells is suitable for port-based network modeling, more precisely in the port-Hamiltonian framework.

Port-Hamiltonian (pH) systems are widely used in modeling, analysis and control of (multi-)physical systems [8], because the representation is based on the energy state space, which represents a natural state space for the equations composing the mathematical models of physical systems.

Port-Hamiltonian systems arise naturally from port-based network modeling, further having a geometric structure. The Hamiltonian gives the total stored energy of the system, whereas port-Hamiltonian systems have boundary ports to interact with the environment, through the exchange of energy. The mathematical representation of a pH system is:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{J}-\mathbf{R})\nabla_{\mathrm{x}}\mathrm{H}(\mathrm{x}) + \mathbf{B}u(t) \\ \mathbf{y} = \mathbf{B}^{\mathrm{T}}\nabla_{\mathrm{x}}\mathrm{H}(\mathrm{x}) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the n-dimensional state vector; $\mathrm{H} : \mathbb{R}^n \to [0, \infty]$ is a scalar-valued vector function (continuously differentiable) – the Hamiltonian, describing the internal energy of the system as a function of state; $\mathbf{J} = -\mathbf{J}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ is the structure matrix describing the interconnection of energy storage elements in the system; $\mathbf{R} = \mathbf{R}^{\mathrm{T}} \geq 0$ is the dissipation matrix describing energy loss in the system; and $\mathbf{B} \in \mathbb{R}^{n \times m}$ is the port matrix describing how energy enters and exits the system through the m terminals.

2.2 Moment-matching based reduction of pH systems

Port-Hamiltonian representations are widely used in lumped parameter system analysis and control, but their framework extends to distributed-parameter and mixed lumped-distributed parameter physical systems. As is most often the case for these complex systems, the state space dimension can be very large, so model reduction is necessary.

Port-Hamiltonian systems exhibit important properties such as passivity which is relevant for stability analysis and they maintain their structure through composition, meaning an interconnection of port – Hamiltonian systems is still port-Hamiltonian [7]. Any reduced order model is expected to retain the properties and structure of the original model. There is extensive research done on model order reduction with preservation of properties and/or port-Hamiltonian structure for linear [4], [1] and nonlinear systems [2], [6].

Among these techniques, the time-domain moment-matching procedure represents an efficient tool [4]. The reduced model is obtained by constructing a lower degree rational function that approximates a given transfer function (assumed rational). The low degree rational function matches the given transfer function at various interpolation points in the complex plane. In [4] a family of models that achieve moment matching is extracted and from this set only the reduced order model that inherits the port-Hamiltonian form is selected.

3 Model reduction

Our approach is based on describing the myelinated compartment in Fig. 2 as a port-Hamiltonian system (1) and reduce the overall model with structure-preserving moment-matching.

We consider the network in Fig 2 as a 2x2 system with input $u = \left[\frac{u_1(t)}{R_1} \;\; i_2(t)\right]^{\mathrm{T}}$ and output $y = [u_{c_1}(t) \;\; u_2(t)]^{\mathrm{T}}$. The state space vector consists of the charges of the capacitors $\mathbf{x} = [q_1, q_2, \ldots, q_n]^{\mathrm{T}}$, thus the derivative $\dot{\mathbf{x}} = [i_{C_1}, i_{C_2}, \ldots, i_{C_n}]^{\mathrm{T}}$ is composed of the currents through the capacitors. The Hamiltonian is defined as:

$$\mathrm{H}(\mathrm{x}) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{C_k}q_k^2 = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \qquad (2)$$

and its derivative with respect to the state variables is a vector of voltages:

$$\nabla_{\mathrm{x}}\mathrm{H}(\mathrm{x}) = [u_{C_1}, u_{C_2}, \ldots, u_{C_n}]^{\mathrm{T}} = \mathbf{Q}\mathbf{x}. \qquad (3)$$

In this formulation, $\mathbf{Q}$ is a diagonal matrix $\mathbf{Q} = \mathrm{diag}\left(\frac{1}{C_k}\right)$, the structure matrix $\mathbf{J} = \mathbf{0}$ and the dissipative matrix $\mathbf{R}$ is a tridiagonal matrix having on line k the elements $-\frac{1}{R_k}$, $\frac{1}{R_k} + \frac{1}{R_{k+1}} + G_k$ and $-\frac{1}{R_k}$. The port matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}^{\mathrm{T}}$.

The efficiency of this reduction will be compared with the results previously obtained in [3], with the prospect of using it for the reduction of the nonlinear global model of a myelinated axon.

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